THE THEORY OF LINEAR THREE-DIMENSIONAL UNSTEADY PROBLEMS OF DIFFRACTION AND CERTAIN NONLINEAR PROBLEMS

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Paper [1] solved in the linear formulation the plane unsteady problem of diffraction of plane weak shock waves by contours of arbitrary shape, in particular by a circle. We give below the generalization of the results [1] to three-dimensional problems of diffraction. Use is made of the following theorem.

Theorem 1. The three-dimensional linear problem of diffraction of plane weak shock waves by bodies of arbitrary shape Q is equivalent to the external problem of flow of a four-dimensional steady supersonic ($M = \sqrt{2}$) stream of ideal gas past a certain hollow semi-infinite four-dimensional cylinder, corresponding to the body Q, at a small angle of attack α_0.

The proof of this theorem is not given - it is analogous to the proof of the corresponding theorem for plane diffraction problems [1].

1. Paper [1] considers the plane linear problem of diffraction of a shock wave by an arbitrary contour C or the diffraction of a weak shock wave by an arbitrary infinite cylinder, when the front of the incident plane shock wave is parallel to the generators of the cylinder, having as its transverse section the contour C.

We shall show that this three-dimensional problem, in the case when the front of the incident shock wave makes with the cylinder axis a certain arbitrary angle γ, reduces to the plane problem of diffraction considered in [1].

Assuming the flow of a plane weak shock wave past an infinite cylinder to be irrotational and isentropic, we can reduce the problem in the dimensionless formulation to finding the perturbation velocity potential Φ, satisfying the equation

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}=\frac{\partial^2\Phi}{\partial \tau^2} \qquad \left(\tau=\frac{at}{l}\right) \tag{1.1}$$

where $2l$ is a characteristic length (for example, the maximum diameter of the cylinder), and the conditions

$$\frac{\partial\Phi}{\partial n}=-\alpha_0\cos\gamma\,\frac{\partial y}{\partial n}+\varepsilon_\tau'(s,z,\tau) \text{ on the cylinder} \tag{1.2}$$

$$\Phi=0 \text{ at the front of the reflected shock wave } S^- \tag{1.3}$$

Here ε is the deformation of the cylinder, n is the normal, s is the tangent to the cylinder in the xy-plane.

The condition (1.2) is derived from the assumption that the potential of the incident shock wave is

$$\Phi_0(x,y,z,\tau)=\alpha_0(y\cos\gamma+z\sin\gamma-\tau) \tag{1.4}$$

and the magnitude of deformation of the cylinder $\varepsilon(s, z, \tau)$ is related to the pressure $p(s, z, \tau)$ on the cylinder by the relationship

$$\varepsilon(s,z,\tau)=k\,[p(s,z,\tau)-p_0] \tag{1.5}$$

where k is a coefficient of proportionality. The solution, which is obtained under the assumption (1.4) and (1.5), is not difficult [1] to generalize to the case when

$$\Phi_0(x,y,z,\tau)=f(y\cos\gamma+z\sin\gamma-\tau), \qquad p(s,z,\tau)=\Psi[\varepsilon(s,z,\tau)]$$

It is not difficult to see that at any τ the front of the incident shock wave intersects the infinite cylinder; the dimensionless velocity of the wave along the z-axis is equal to cosec γ.

Let us introduce a moving system of coordinates

$$x=x, \qquad y=y, \qquad \eta=\tau\sec\gamma-z\tan\gamma \tag{1.6}$$

In this system of coordinates the potential $\Phi(x, y, z, \tau)$ must depend only on x, y, η. Similarly $\varepsilon(s, z, \tau)$ also will have the form

$$\varepsilon(s,z,\tau)=\varepsilon(s,\eta) \tag{1.7}$$

Moreover it is evident that when $\eta\leqslant 0$ the potential $\Phi\equiv 0$.

In the coordinates (1.6) equation (1.1) and conditions (1.2) to (1.5) assume the forms

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = \frac{\partial^2\Phi}{\partial \eta^2} \tag{1.8}$$

$$\frac{\partial\Phi}{\partial n} = -\alpha_0 \cos\gamma \frac{\partial y}{\partial n} + \lambda_1 \frac{\partial^2\Phi}{\partial\eta^2} \quad \text{on } C \qquad \left(\lambda_1 = -\frac{\rho a^2 k}{\cos^2\gamma}\right) \tag{1.9}$$

$$\Phi = 0 \quad \text{when } \eta \leqslant 0 \quad \text{on } S^- \tag{1.10}$$

$$\Phi_0(x, y, \eta) = \alpha_0 \cos\gamma\, (y - \eta), \qquad \varepsilon(s, \eta) = k\,[p(s, \eta) - p_0] \tag{1.11}$$

Here (1.9) takes into account that $p(s, \eta) \sim \partial\Phi/\partial\eta$. The system (1.8) to (1.10) is equivalent to the plane problem of diffraction [1] of a shock wave by a contour C with the only difference that in place of τ we have η, and the dimensionless constants depend on γ.

For example, for a circular cylinder of unit radius, when $\eta > 0$ the potential $\Phi(x, y, z, \tau)$ at the surface of the cylinder satisfies the integro-differential equation

$$\Phi(1, \theta_0, \eta_0) = \frac{1}{2\pi} \frac{\partial}{\partial\eta_0} \left\{ \iint_\Sigma \left[\Phi(1, \theta, \eta) \frac{\partial V}{\partial r} - \frac{\partial\Phi}{\partial r} V \right] d\theta\, d\eta \right\} \tag{1.12}$$

where Σ is the part of the surface of the cylinder in the auxiliary plane diffraction problem [1], cut off by the cone of influence from the point $(r_0 = 1, \theta_0, \eta_0)$; r and θ are polar coordinates in the (x, y) plane; V is the Volterra function of the three-dimensional wave equation.

For large η or far behind the front of the incident shock wave

$$\Phi(r_0, \theta_0, z_0, \tau_0) = \frac{1}{2\pi i} \int_M \frac{K_1(r_0 q) \exp[q(\tau_0 \sec\gamma - z_0 \tan\gamma)]}{K_1'(q) + \lambda_1 q K_1(q)} \frac{dq}{q^2} \tag{1.13}$$

Here M is the contour of integration of the inverse Laplace transformation, $K_1(q)$ is the Bessel function with imaginary argument.

The deformation ε at large η, for example, is obtained without difficulty as

$$\begin{aligned} \varepsilon(\theta, z, \tau) = \lambda_1\alpha_0 \cos\gamma \sin\theta \Big\{ &\exp[-q_1(\lambda_1)(\tau\sec\gamma - z\tan\gamma)] \times \\ \times [A(\lambda_1)\cos\{q_2(\lambda_1)(\tau\sec\gamma - z\tan\gamma)\} &+ B(\lambda_1)\sin\{q_2(\lambda_1)(\tau\sec\gamma - z\tan\gamma)\}] - \\ - \int_0^\infty \frac{\exp[-q(\tau\sec\gamma - z\tan\gamma)]\, dq}{\{[K_1'(q) + \lambda_1 q K_1(q)]^2 + \pi^2[I_1'(q) + \lambda_1 q I_1(q)]^2\}\, q^2} \Big\} \end{aligned} \tag{1.14}$$

Here $q_1(\lambda_1)$, $q_2(\lambda_1)$, $A(\lambda_1)$ and $B(\lambda_1)$ are completely determinate functions of λ_1 and $-q_1(\lambda_1) + iq_2(\lambda_1)$ is the root of the equation

$$K_1'(q) + \lambda_1 q K_1(q) = 0 \qquad (q_1(\lambda_1) > 0)$$

2. Let us consider the problem of diffraction of a weak shock wave by a sphere of unit radius (Fig. 1). In its dimensionless formulation, the problem reduces to determination of the potential Φ of the perturbation velocity according to the equation

$$\frac{\partial^2\Phi}{\partial r^2} + \frac{2}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) = \frac{\partial^2\Phi}{\partial\tau^2} \qquad (2.1)$$

in the region between the surface of the sphere and the reflected shock wave, with the conditions

$$\partial\Phi/\partial r = -\alpha_0\cos\theta + \varepsilon_\tau'(\theta, \tau) \quad \text{when } r = 1$$

$$\Phi = 0 \quad \text{when } \tau \leqslant 0 \qquad (2.2)$$

$$\Phi = 0 \quad \text{on the reflected shock wave } S^- \qquad (2.3)$$

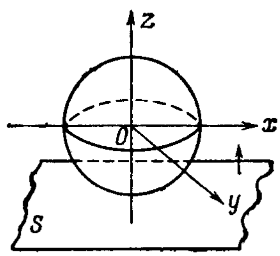

Fig. 1.

Here r, θ and φ are spherical coordinates, i.e.

$$x = r\cos\varphi\sin\theta, \qquad y = r\sin\varphi\sin\theta, \qquad z = r\cos\theta$$

By virtue of symmetry the potential Φ does not depend on φ.

Just as in Section 1, we assume that the potential Φ_0 of the incident shock wave is equal to

$$\Phi_0(x, y, z, \tau) = \alpha_0(z - \tau)$$

and the deformation of the sphere $\varepsilon(\theta, \tau)$ is linearly related to the pressure $p(\theta, \tau)$ on the sphere according to formula (1.5). According to Theorem 1 we shall consider the auxiliary external problem of flow of a supersonic stream of ideal gas at a small angle of attack α_0 past a four-dimensional hollow cylinder which is semi-infinite with respect to the τ-axis ($\tau \geqslant 0$).

Let us consider the pattern of diffraction of the shock wave by the sphere at a certain moment of time $\tau_1 > 0$. At $\tau = \tau_1$ the boundary of the perturbed region in the diffraction problem consists of the reflected shock wave and part of the surface of the sphere, when $\tau_1 < \tau_2$, or of the whole surface of the sphere when $\tau_1 \geqslant \tau_2$, where τ_2 is the time at which the incident shock wave completely envelopes the sphere. Let us now cut the surface of the four-dimensional cylinder by the plane $\tau = \tau_1$. Then the four-dimensional curve of the intersection of the cylinder by the plane $\tau = \tau_1$ will correspond to the part of the surface of the sphere (or the whole surface of the sphere when $\tau_1 \geqslant \tau_2$), comprising the boundary of the perturbed region in the diffraction problem.

By virtue of the theorem mentioned above, the external auxiliary problem is likewise described by the system (2.1) to (2.3), which we shall solve by a method analogous to Volterra's method for the three-dimensional wave equation.

Let us construct the solution $V(r, \theta, \varphi, \tau; r_0, \theta_0, \varphi_0, \tau_0)$ of the wave equation (2.1) which vanishes at the surface of the characteristic cone

$$(\tau_0 - \tau)^2 - (x_0 - x)^2 - (y_0 - y)^2 - (z_0 - z)^2 = 0 \tag{2.4}$$

It is not difficult to see that the function V has the form [2]

$$V = \frac{\sqrt{(x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2} - (\tau_0 - \tau)}{\sqrt{(x_0 - x^2) + (y_0 - y)^2 + (z_0 - z)^2}} \tag{2.5}$$

Following the method of solution of the wave equation with several independent variables [2], we can show that

$$\Phi(r_0, \theta_0, \tau_0) = \frac{1}{4\pi} \frac{\partial^2}{\partial \tau_0^2} \left\{ \iiint_T \left[\Phi(1, \theta, \tau) \frac{\partial V}{\partial r} - \frac{\partial \Phi}{\partial r} V \right] d\theta \, d\varphi \, d\tau \right\} \tag{2.6}$$

Since the value of Φ on the surface of the four-dimensional cylinder is unknown, we let r_0 tend to 1 in (2.6) and obtain an integro-differential equation for

$$\Phi(1, \theta_0, \tau_0) = \frac{1}{4\pi} \frac{\partial^2}{\partial \tau_0^2} \left\{ \iiint_{T_0} \left[\Phi(1, \theta, \tau) \frac{\partial V}{\partial r} - \frac{\partial \Phi}{\partial r} V \right] d\theta \, d\varphi \, d\tau \right\} \tag{2.7}$$

where T and T_0 are the volumes or parts of the surfaces of the four-dimensional cylinder cut off by the cones of influence (2.4) from the points $(r = r_0, \theta_0, \tau_0, \varphi_0)$ and $(r_0 = 1, \theta_0, \tau_0, \varphi_0)$ (Fig. 2), respectively. Determining $\Phi(1, \theta, \tau)$ from equation (2.7), we find the quantities $p(1, \theta, \tau)$ and $\varepsilon(\theta, \tau)$ according to the formulas

$$p(1, \theta, \tau) = p_0 - \rho a^2 \frac{\partial \Phi}{\partial \tau}, \qquad \varepsilon(\theta, \tau) = \lambda \frac{\partial \Phi}{\partial \tau}, \qquad \lambda = -\rho a^2 k \tag{2.8}$$

We can give an asymptotic solution of the problem of diffraction by a sphere for large τ. In fact, for large τ the radius r of the reflected shock wave depends weakly on θ and the potential Φ can be sought in the form

$$\Phi(r, \theta, \tau) = -\alpha_0 \cos\theta f(r, \tau) \tag{2.9}$$

Let us substitute (2.9) in (2.1) and to the equation so obtained for

$f(r, \tau)$ let us apply the Laplace transformation. For the transformed function $F(r, q)$, where q is the variable of the Laplace transformation, we obtain Bessel's equation

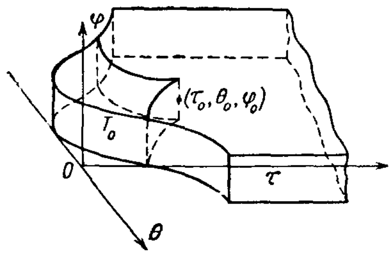

Fig. 2.

$$\frac{d^2F}{dr^2} + \frac{2}{r}\frac{dF}{dr} - \left(q^2 + \frac{2}{r^2}\right)F = 0 \quad (2.10)$$

under the condition that

$$\frac{dF}{dr} = \frac{1}{q} - \lambda q^2 F \quad \text{when } r = 1 \quad (2.11)$$

and the function $f(r, \tau)$, as well as its derivative with respect to τ, vanish when $\tau \leqslant 0$.

It is not difficult to see that

$$F(r, q) = \frac{1}{\sqrt{r}}\frac{K_{3/2}(rq)\, q^{-2}}{K_{3/2}'(q) + \lambda q K_{3/2}(q)}, \qquad f(r, \tau) = \frac{1}{2\pi i}\int_M F(r, q)\, e^{q\tau}\, dq \quad (2.12)$$

where $K_{3/2}(q)$ is the spherical Bessel function with imaginary argument.

Substituting (2.12) in (2.9), we obtain an expression for $\Phi(r, \theta, \tau)$ for large τ, after which we can calculate the pressure p and the deformation ε of the cylinder.

When $\lambda = 0$ or $k = 0$ we obtain the solution of the problem of diffraction by a rigid sphere.

3. Let us consider the diffraction of a shock wave by a prolate ellipsoid of revolution. Let us introduce spheroidal coordinates, setting

$$x = \frac{a_0}{2}\sqrt{(\xi^2 - 1)(1 - \eta^2)}\cos\varphi$$
$$y = \frac{a_0}{2}\sqrt{(\xi^2 - 1)(1 - \eta^2)}\sin\varphi$$
$$z = \frac{a_0}{2}\xi\eta$$

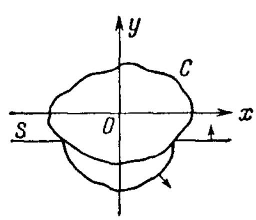

Fig. 3.

where a_0 is the dimensionless focal distance and we shall suppose that the shock wave is propagating along the z-axis in the positive direction.

Then again by virtue of the symmetry of the problem the potential Φ satisfyin the wave equation

$$\frac{4}{a_0^2(\xi^2 - \eta^2)}\left\{\frac{\partial}{\partial\xi}\left[(\xi^2 - 1)\frac{\partial\Phi}{\partial\xi}\right] + \frac{\partial}{\partial\eta}\left[(1 - \eta^2)\frac{\partial\Phi}{\partial\eta}\right]\right\} = \frac{\partial^2\Phi}{\partial\tau^2} \quad (3.1)$$

will not depend on φ, whilst Φ satisfies the conditions

$$\frac{\partial\Phi}{\partial\xi} = -\frac{\alpha_0 a_0 \eta}{2} + \varepsilon'(\eta, \tau) \quad \text{when } \xi = 1, \qquad \Phi = 0 \quad \text{when } \tau \leqslant 0 \text{ and on } S^- \tag{3.2}$$

The surface of the ellipsoid corresponds to the value $\xi = 1$.

By virtue of Theorem 1, as in Section 2, we shall solve the corresponding auxiliary problem; when $\xi = 1$ we can obtain an integro-differential equation for Φ of the form

$$\Phi(1, \eta_0, \tau_0) = \frac{1}{4\pi}\frac{\partial^2}{\partial\tau_0^2}\left\{\iiint\limits_{T_1}\left[\Phi(1, \eta, \tau)\frac{\partial V}{\partial\xi} - \frac{\partial\Phi}{\partial\xi}V\right]d\eta\, d\varphi\, d\tau\right\} \tag{3.3}$$

For an oblate ellipsoid of revolution the equation for Φ when $\xi = 1$ will have almost the same form as equation (3.3), with just the difference that a_0 will be replaced by $-ia_0$ and $\xi = \cosh\mu$ by $i\sinh\mu$.

Note. The asymptotic solution for large τ of the three-dimensional diffraction problem can be obtained only for a sphere (or for a circle in plane problems).

4. Let us consider the plane nonlinear problem of diffraction of a weak shock wave round a contour C of arbitrary shape (Fig. 3).

We shall assume that the field of flow behind the front of the shock wave is irrotational and isentropic, which is valid only up to quantities of the second order in the intensity of the shock. Let us introduce the velocity potential Φ_1 of the plane flow past the contour C. Then up to third order forms the potential $\Phi = \Phi_1/al$ satisfies the equation

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} - \frac{\partial^2\Phi}{\partial\tau^2} = (n_0 - 1)\frac{\partial\Phi}{\partial\tau}\frac{\partial^2\Phi}{\partial\tau^2} + 2\left(\frac{\partial\Phi}{\partial x}\frac{\partial^2\Phi}{\partial x\,\partial\tau} + \frac{\partial\Phi}{\partial y}\frac{\partial^2\Phi}{\partial y\,\partial\tau}\right) \tag{4.1}$$

where a is the velocity of sound, l is a characteristic linear parameter of the problem, n_0 is the adiabatic index, x and y are dimensionless coordinates.

The boundary conditions for Φ are expressed by:

1) no flow of gas through the given contour C,

2) continuity of potential at the front of the reflected shock wave.

Let us consider the first boundary condition.

It means that the velocity of particles across the contour C is zero, i.e.

$$\partial\Phi / \partial n = 0 \qquad \text{on } C \tag{4.2}$$

Let us denote by $\Phi_0(x, y, \tau)$ the velocity potential of the gas behind the shock when $\tau < 0$. Then the second boundary condition can be written in the form

$$\Phi(x, y, \tau) = \Phi_0(x, y, \tau) \quad \text{on } S^- \text{ when } \tau > 0 \tag{4.3}$$

and equation (4.3) becomes an identity when $\tau \leqslant 0$. For simplicity we shall assume that

$$\Phi_0(x, y, \tau) = \alpha_0 (y - \tau), \qquad \alpha_0 = \frac{\Delta p}{\rho a^2} \tag{4.4}$$

where Δp is the increase in pressure at the front of the incident shock wave, ρ is the density of the unperturbed gas. Let us set

$$\Phi(x, y, \tau) = \Phi_0(x, y, \tau) + \varphi_1(x, y, \tau) + \varphi_2(x, y, \tau) + \ldots \tag{4.5}$$

where φ_i is a quantity of the ith order of smallness. Substituting (4.5) in (4.1), we obtain

$$\frac{\partial^2\varphi_1}{\partial x^2} + \frac{\partial^2\varphi_1}{\partial y^2} - \frac{\partial^2\varphi_1}{\partial \tau^2} = 0$$

$$\frac{\partial\varphi_1}{\partial n} = -\alpha_0 \frac{\partial y}{\partial n} \quad \text{on } C, \qquad \varphi_1 = 0 \quad \text{when } \tau \leqslant 0 \text{ and on } S^- \tag{4.6}$$

$$\frac{\partial^2\varphi_2}{\partial x^2} + \frac{\partial^2\varphi_2}{\partial y^2} - \frac{\partial^2\varphi_2}{\partial \tau^2} = (n_0 - 1)\frac{\partial\varphi_1}{\partial\tau}\frac{\partial^2\varphi_1}{\partial\tau^2} + 2\left(\frac{\partial\varphi_1}{\partial x}\frac{\partial^2\varphi_1}{\partial x\,\partial\tau} + \frac{\partial\varphi_1}{\partial y}\frac{\partial^2\varphi_1}{\partial y\,\partial\tau}\right) \tag{4.7}$$

$$\frac{\partial\varphi_2}{\partial n} = 0 \quad \text{on } C, \qquad \varphi_2 = 0 \quad \text{when } \tau \leqslant 0 \text{ and on } S^- \tag{4.8}$$

In this way, the problem of diffraction is reduced to solution of the systems (4.6) and (4.7) to (4.8) for the potentials $\varphi_1(x, y, \tau)$ and $\varphi_2(x, y, \tau)$.

The linear diffraction problem (4.6) has been solved [1], so we shall solve the diffraction problem in the second approximation.

Let us proceed to the solution of the system (4.7) to (4.8).

As in the solution of the linear diffraction problem [1] (i.e. in the solution of the system (4.6)) we can show that the following theorem holds.

Theorem 2. The problem of diffraction of a weak shock wave by a contour C in the second approximation (4.7) to (4.8) is equivalent to the auxiliary external problem of flow of a supersonic ($M = \sqrt{2}$) steady stream of ideal gas past a hollow cylinder which is semi-infinite along the τ-axis ($\tau \geqslant 0$) [1] at a small angle of attack α_0 in the second approximation, satisfying the same system (4.7) to (4.8) with

$2n_1 = n_0 - 1$, where n_1 is the adiabatic index in the auxiliary problem.

We notice that the curve of intersection of the surface of the hollow cylinder with the plane $\tau = \tau_1$ corresponds to a part of the contour C when $\tau_1 < \tau_2$, comprising the boundary of the reflected shock, or to the whole contour C when $\tau_1 \geqslant \tau_2$, where τ_2 is the time at which the incident shock wave completely engulfs the contour C. Accordingly, we shall solve the auxiliary external problem, satisfying the system (4.7) to (4.8), whilst the perturbations from the internal surface will be neglected.

With supersonic velocities of flow past the hollow cylinder, the problem of finding the solution of equation (4.7) is complicated by the fact that the term standing on the right-hand side of this equation undergoes a discontinuity close to the surface of the cylinder. This is due to the fact that the solution of the linearized equation (4.6) undergoes a discontinuity on C, since the boundary conditions for $\varphi_1(x, y, \tau)$ are given on C. In the solution of equation (4.7) the boundary conditions for $\varphi_2(x, y, \tau)$ must also be given on C, which must not lie in the field of flow. It is evident that the curve C lies outside the field of flow (it is its boundary) and therefore, in principle, the system (4.7) to (4.8) is soluble.

The system (4.7) to (4.8) will be solved by Volterra's method. In a similar way we obtain

$$\varphi_2(x, y, \tau) = \frac{1}{2\pi} \frac{\partial}{\partial \tau_0} \left\{ \iint\limits_{\Sigma} \varphi_2(x, y, \tau) \frac{\partial V}{\partial n} \, ds \, d\tau + \iiint\limits_{T} F(\xi, \eta, \zeta) V \, d\xi \, d\eta \, d\zeta \right\} \tag{4.9}$$

Here $F(x, y, \tau)$ is the right-hand side of equation (4.7); Volterra's function has the form

$$V = \ln \frac{(\tau_0 - \tau) + \sqrt{(\tau_0 - \tau)^2 - (x_0 - x)^2 - (y_0 - y)^2}}{\sqrt{(x_0 - x)^2 + (y_0 - y)^2}} \tag{4.10}$$

The integration in (4.9) is carried out with respect to the part of the surface Σ of the cylinder which is cut off by the cone of influence

$$(\tau_0 - \tau)^2 - (x_0 - x)^2 - (y_0 - y)^2 = 0 \tag{4.11}$$

from the arbitrary point (x_0, y_0, τ_0) and with respect to the volume T included inside the cone of influence (4.11) and bounded by the surface of the cylinder Σ, by the part of the wave surface cut off by the cone of influence (4.11), and by the surface of the cone of influence [1].

Letting the point (x_0, y_0, τ_0) tend to the surface of the cylinder in equation (4.9), we obtain an integral equation for the potential φ_2 on the surface of the cylinder.

In the solution of the problem (4.7) to (4.8) it is assumed that the contour is not deformable.

Let us assume that the contour is deformable and its deformation ε depends linearly on the pressure $p(s, \tau)$ on C, i.e.

$$\varepsilon(s, \tau) = k[p(s, \tau) - p_0] \tag{4.12}$$

where k is the coefficient of proportionality and s is the arc length. The condition (4.2) takes the form

$$\frac{\partial \Phi}{\partial n} = \varepsilon_\tau'(s, \tau) \quad \text{on } C \tag{4.13}$$

Let us set

$$\varepsilon(s, \tau) = \varepsilon_1(s, \tau) + \varepsilon_2(s, \tau) + \ldots = k[p_1(s, \tau) + p_2(s, \tau) + \ldots - p_0] \tag{4.14}$$

If the deformation $\varepsilon_1(s, \tau)$ is known [1], then equation (4.9) takes the form

$$\varphi_2(x_0, y_0, \tau_0) = \frac{1}{2\pi}\frac{\partial}{\partial \tau_0}\left\{\iint\limits_{\Sigma}\left[\varphi_2(x, y, \tau)\frac{\partial V}{\partial n} - \frac{\partial \varphi_2}{\partial \nu}V\right]ds\, d\tau + \iiint\limits_{T} F(\xi, \eta, \zeta)\, V\, d\xi\, d\eta\, d\zeta\right\} \tag{4.15}$$

where

$$\lambda = -\rho a^2 k, \qquad \frac{\partial \varphi_2}{\partial \nu} = \lambda\left(\frac{\partial^2 \varphi_2}{\partial \tau^2} - \frac{\partial \varphi_1}{\partial \theta}\frac{\partial^2 \varphi_1}{\partial \theta\, \partial \tau} + \frac{\partial \varphi_1}{\partial \tau}\frac{\partial^2 \varphi_1}{\partial \tau^2}\right) \tag{4.16}$$

In the particular case when the contour C is a circle, the equations (4.9) and (4.15) have the same form. For a circle, moreover, we can construct the asymptotic solution of the system (4.7) to (4.8), where one of the boundary conditions can be replaced by the condition (4.13). By the same token, for large τ we can set

$$\varphi_2(x, y, \tau) = f_1(r, \tau) + \cos 2\theta f_2(r, \tau) \tag{4.17}$$

where r and θ are polar coordinates.

Let us substitute (4.17) in (4.7), (4.8) and (4.13) and to the system so obtained let us apply the Laplace transformation. Then $F_1(r, q)$ and $F_2(r, q)$ will satisfy equations

$$\frac{d^2 F_1}{dr^2} + \frac{1}{r}\frac{dF_1}{dr} - q^2 F_1 = \Psi_0(r, q), \qquad \frac{d^2 F_2}{dr^2} + \frac{1}{r}\frac{dF_2}{dr} - \left(q^2 + \frac{4}{r^2}\right)F_2 = \Psi_2(r, q) \tag{4.18}$$

$$\frac{dF_1}{dr} = \lambda q^2 F_1, \qquad \frac{dF_2}{dr} = \lambda q^2 F_2 \quad \text{when } r = 1 \tag{4.19}$$

The function $\varphi_2(x, y, \tau)$ and its derivatives with respect to τ must vanish when $\tau \leqslant 0$. The functions F_1, F_2 and Ψ_0, Ψ_2 are the transforms of the functions f_1, f_2 and ψ_0, ψ_2, obtained by the Laplace transformation. It is assumed here that

$$F(r, \theta, \tau) = \psi_0(r, \tau) + \cos 2\theta \psi_2(r, \tau), \qquad \varphi_1(r, \theta, \tau) - \alpha_0 \sin\theta f_1(r, \tau) \tag{4.20}$$

The function $f_1(r, \tau)$ is assumed to be known [1].

It is not difficult to see that the solution of equations (4.18) has the form

$$\begin{aligned} F_1(r, q) &= A_0(r, q) K_0(rq) + B_0(r, q) I_0(rq) \\ F_2(r, q) &= A_2(r, q) K_2(rq) + B_2(r, q) I_2(rq) \end{aligned} \tag{4.21}$$

Here $I_i(rq)$ and $K_i(rq)$ ($i = 0$ or 2) are the Bessel functions of imaginary argument and

$$A_i(r, q) = -\left\{\int_r^\infty r\Psi_i(r, q) I_i(rq)\, dr + \frac{I_i'(q)}{K_i'(q) - \lambda q K_i(q)} \int_1^\infty r\Psi_i(r, q) I_i(rq)\, dr - \right.$$

$$\left. - \frac{K_i(q)}{K_i'(q) - \lambda q K_i(q)} \int_1^\infty r\Psi_i(r, q) K_i\, dr\right\}, \quad B_i(r, q) = -\int_r^\infty r\Psi_i(r, q) K_i(rq)\, dr \tag{4.22}$$

Substituting (4.20) in (4.17), we obtain

$$\varphi_2(r, \theta, \tau) = \frac{1}{2\pi i} \int_M [F_1(r, q) + \cos 2\theta F_2(r, q)] e^{q\tau}\, dq \tag{4.23}$$

From the value of $\varphi_2(r, \theta, \tau)$ so obtained and the known $\varphi_1(r, \theta, \tau)$ let us determine $p(\theta, \tau)$ on the contour and the deformation $\varepsilon(\theta, \tau)$ according to the formulas

$$p(\theta, \tau) = p_0 - \rho a^2 \left(\frac{\partial \varphi_1}{\partial \tau} + \frac{\partial \varphi_2}{\partial \tau}\right), \qquad \varepsilon(\theta, \tau) = \lambda \left(\frac{\partial \varphi_1}{\partial \tau} + \frac{\partial \varphi_2}{\partial \tau}\right) \tag{4.24}$$

By virtue of the equivalence of the problems the formulas (4.15) and (4.23) give the solution of the problem of diffraction of a shock wave by a circle to the second approximation. Formula (4.15) gives the solution of the diffraction problem also for an arbitrary contour C.

Note. It can be shown that the three-dimensional diffraction problem can be solved to the second approximation by a method similar to that described above.

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